# lecture #01

## Introduction (Real and imaginary numbers)

### Topic #01:-

## Imaginary Number:
## An introduction and Applications in daily life.

- Natural and Rational Number are
  Imaginary number:-
  Example:- $1/2$, $3/4$, $100$ —
- Irrational Number => "$\pi$" —
- Signal processing (Crowds -> wesipeig)
- (AC) Imaginary ~~number~~ currents
- $\boxed{Z_{new} = Z^2_{odd} + \text{Constant}}$ => Algorithm.

### Topic #02:-

"$i$" = $90°$

## Real Number and their Transformation 1

In early ages Number and points are consider as different thing. Numbers are studied in Algebra nd points are studied in Geometric.

But in present era:-

"Points are nothing but the numbers and numbers are nothing but the points" on the real line".

* Addition is a translation on a real line number
* Multiplication:- ( scaling )

Contraction ↓ (making small)

dilation ↓ (lengthen)

**MCQ'S**

(-1) is Half rotation (180°)

$(-1)^2 \rightarrow$ Complete rotation (360°)

## Topic :- 03

~~Real Number~~

### Imaginary Numbers (definition and its geometrical representation)

The Numbers on 90° rotation on a real line will be above or blew the real line and that numbers only can be imagin

90° rotation will correspond to imaginary number.

## imaginary numbers:-

"Imaginary numbers are nothing but there are a quarter rotation of any number which will transform 90° rotation of any number."

$$\sqrt{-1} = 'i'$$

## Topic :- 04

### Integral power of Iota

$i = \sqrt{-1}$

$i^2 = (i \cdot i) = -1$

$i^3 = \cancel{\quad} -1$

$i^4 = 1$

$i^5 = i$

$i^{2n} = (i^2)^n = (-1)^n$

$$= \begin{cases} 1 & n \in E \\ -1 & n \in O \end{cases}$$

Mutiplicative inverse = Additive inverse

## Caution (regarding Imaginary number)

**Caution :-** $x, y \in \mathbb{R}^+$

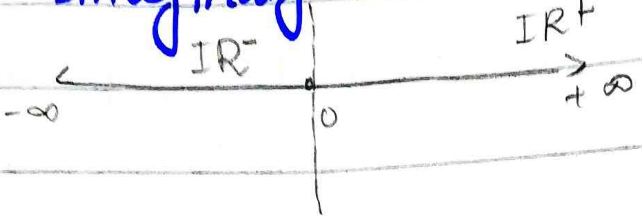
$\sqrt{x}\sqrt{y} = \sqrt{xy} \Rightarrow$ real Number

But in '$i$'

$i = \sqrt{-1}$

$\sqrt{-2} = \sqrt{(-1)(2)} = \sqrt{2}\, i$

## Sign of Imaginary Number :-




if $i > 0$

$\Rightarrow i \cdot i > i(0)$

$\Rightarrow i^2 > 0$

$\Rightarrow (-1) > 0$ ( a contradiction)

if $i < 0$

$\Rightarrow i \cdot i < i(0)$

$\Rightarrow i^2 < 0$

$\Rightarrow (-1) < 0 \Rightarrow$ a contradiction

Square of any number Can't be less-than Zero

"therefore '$i$' has no sign"

# Complex number, their algebra, polar and Argand Diagram

Lecture # 02:-

## * Complex Numbers:-

A complex number is a Combination of real and Imaginary number.

$$\underbrace{7}_{\text{Real}} + \underbrace{3i}_{\text{Imaginary}}$$

## * Argand diagram [Complex number]

A Collection of all the Complex number is call argand ~~dig~~ diagram.

## * Algebra of Complex Number:-

(i) Equality :- $Z \cong \mathbb{R}\times\mathbb{R}$

for example we have two complex number.

$$z_1 = x_1 + iy_1 , \quad z_2 = x_2 + iy_2$$

$$z_1 = z_2 \iff x_1 + iy_1 = x_2 + iy_2$$

$$\left\{ (x_1, y_1) = (x_2, y_2) \right.$$

$$\Rightarrow x_1 = x_2 , \quad y_1 = y_2$$

$$Re\, z_1 = Re\, z_2 , \quad Im\, z_1 = Im\, z_2$$

### • Addition and Subtaraction of Complex number

$$z_1 = x_1 + iy_1 , \quad z_2 = x_2 + iy_2$$

$$z_1 + z_2 = (x_1 + iy_1) + (x_2 + iy_2)$$

$$= (x_1 + x_2) + i(y_1 + y_2)$$

$$z_1 - z_2 = (x_1 + iy_1) - (x_2 + iy_2)$$
$$= (x_1 - x_2) \pm i(y_1 - y_2)$$

## * Conjugate of a Complex Number:-

Conjugate is nothing but the reflection of any complex Number about its reflexsive ~~origion~~ position

$$z = x + iy$$
$$\bar{z} = x - iy$$

## * Multiplication, Division and Related laws:-

- **Mutiplication:-**

$$z_1 \cdot z_2 = (x_1 + iy_1) \cdot (x_2 + iy_2)$$
$$= x_1 x_2 + i x_1 y_2 + i x_2 y_1 + i^2 y_1 y_2$$
$$= x_1 x_2 + i(x_1 y_2 + x_2 y_1) + (-1) y_1 y_2$$
$$= (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1)$$

- **Division:-**

$$z_1 = x_1 + iy_1, \quad z_2 = x_2 + iy_2$$

$$\frac{z_1}{z_2} = \frac{x_1 + iy_1}{x_2 + iy_2}$$

$$= \frac{x_1 + iy_1}{x_2 + iy_2} \times \frac{x_2 - iy_2}{x_2 - iy_2}$$

$$= \frac{x_1x_2 - ix_1y_2 + ix_2y_1 - i^2y_1y_2}{(x_2)^2 - (iy_2)^2}$$

$$= \frac{x_1x_2 - i(x_1y_2 + x_2y_1) - (-1)y_1y_2}{(x_2)^2 - (-1)y_2^2}$$

$$= \frac{(x_1x_2 + y_1y_2) + i(x_1y_2 + x_2y_1)}{x_2^2 + y_2^2}$$

$$= \frac{x_1x_2 + y_1y_2}{x_2^2 + y_2^2}, + i\left(\frac{x_1y_2 + x_2y_1}{x_2^2 + y_2^2}\right)$$

## * Related law's:-

$\cancel{z_1 z_2 = z_2 z_1,}$

(i) Commutative law:-

$$z_1 z_2 = z_2 z_1 \quad , \quad z_1 + z_2 = z_2 + z_1$$

(ii) Associative law:-

$$z_1.(z_2 z_3) = (z_1 z_2).z_3$$

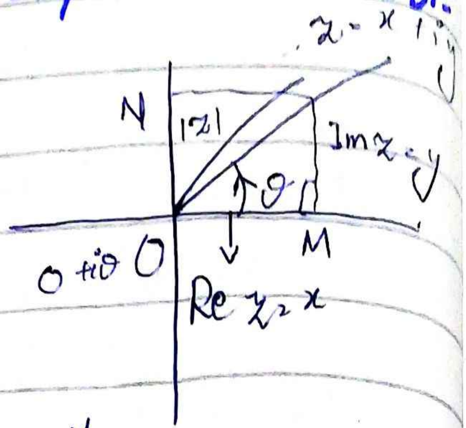
$$z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3$$

(iii) Distributive law:-

$$z_1(z_2 + z_3) = z_1z_2 + z_1z_3$$

# • Polar form of a Complex number...

$z = x + iy$




form ⊥ ΔOPM

$\cos\theta = \frac{OM}{OP} = \frac{Re\, z}{|z|} = \frac{x}{|z|}$

$x = |z| \cos\theta \rightarrow (1)$

$\sin\theta = \frac{PM}{OP} = \frac{Im\, z}{|z|} = \frac{y}{|z|}$

$y = |z| \sin\theta \; (2)$

$\therefore z = x + iy = |z| \cos\theta + i|z| \sin\theta$

$= z(\cos\theta + i\sin\theta)$

$z = z \cos\theta \rightarrow (3)$ polar form of 'z'.

from ⊥ ΔOPM, applying Pythagoem theorem

$|z|^2 = \overline{OP}^2 = \overline{OM}^2 + \overline{PM}^2 = x^2 + y^2 =$

$|z|^2 = x^2 + y^2$

$|z| = \sqrt{x^2 + y^2} = \sqrt{(Re\, z)^2 + (Im\, z)^2}$

from ⊥ OPM

$\tan\theta = \frac{PM}{OM} = \frac{Im\, z}{Re\, z} = \frac{y}{x}$

$\theta = \tan^{-1} \frac{y}{x}$

$\boxed{Argu = \theta = \tan^{-1} \frac{y}{x}}$

## Uniqueness of polar form and principle Arguments:-

$\theta$ or $\theta \pm \xi \kappa(2\pi)$, $\kappa \in \mathbb{Z}$ have same orientation.

$$z = |z|(\cos\theta + i\sin\theta) = |z|(\cos(\theta + 2k\pi) + i\sin(\theta + 2k\pi))$$

$$k \in \mathbb{Z}$$

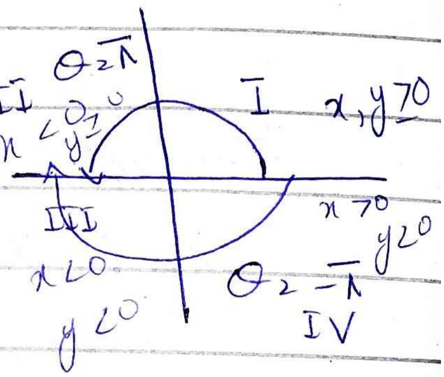
$$\arg z = \theta = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{y}{x}\right) \pm 2k\pi$$

$$-\pi < \theta < \pi \pm 2k\pi$$

## Evalution of principal Arguments:-

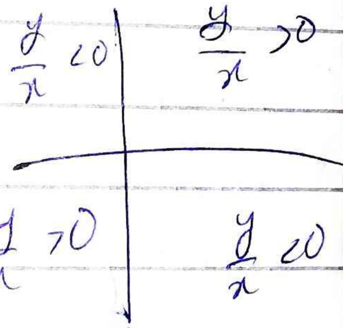
$$z = x + iy$$

$$-\pi < \text{Argu } z \leq \pi$$




$$\text{Arg } z = \theta = \tan^{-1}\left(\frac{y}{x}\right)$$

$$\tan\theta = \frac{y}{x}$$

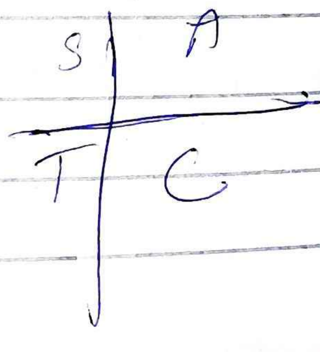



$z = -i + \sqrt{3}i$

$x = -1$, $y = \sqrt{3} > 0$

$< 0$

Terminal ray of '$z$' lies in IInd quad.



$Arg\, z = \pi + \tan^{-1}\left(\frac{y}{x}\right) = \pi + \tan^{-1}\left(\frac{\sqrt{3}}{-1}\right)$

$= \pi + \tan^{-1}(-\sqrt{3})$

$\left\{ \begin{array}{l} \tan^{-1}(-\theta) = -\tan^{-1}\theta \\ \tan^{-1}(\sqrt{3}) = 60^\circ = \pi/3 \end{array} \right.$

$Arg\, z = \pi - \tan^{-1}(\sqrt{3}) = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$

Now; $z = |z|(\cos\theta + i\sin\theta)$ — (i)

$|z| = \sqrt{x^2+y^2} = \sqrt{(-1)^2+(\sqrt{3})^2} = 2$

$z = 2\left(\cos\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)$

$= 2\, cis\,(2\pi/3)$

## Product of Complex Numbers in Polar form:-

$z_1 = r_1\, cis\,\theta_1$, $z_2 = r_2\, cis\,\theta_2$

$r_1 = |z_1|$, $r_2 = |z_2|$

$z_1 z_2 = (r_1\, cis\,\theta_1)(r_2\, cis\,\theta_2)$

$= r_1 r_2(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)$

$= r_1 r_2[\cos\theta_1\cos\theta_2 + i\cos\theta_1\sin\theta_2 + i\sin\theta_1\cos\theta_2 + i^2\sin\theta_1\sin\theta_2]$ $\because i^2 = -1$

$= r_1 r_2[(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)]$

$\because \cos\alpha \cdot \cos\beta - \sin\alpha \sin\beta = \cos(\alpha+\beta)$

$\because \sin\alpha \cos\beta + \cos\alpha \sin\beta = \sin(\alpha+\beta)$

$$z_1 z_2 = r_1 r_2 \left[\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)\right]$$

By induction:-

$$z_1 \cdot z_2 \cdots z_n = r_1 r_2 \cdots r_n \left[(\cos(\theta_1+\theta_2+\cdots\theta_n) + i\sin(\theta_1+\theta_2+\cdots\theta_n)\right]$$

## Division of Complex Numbers in Polar form:-

$z_1 = r_1 \operatorname{cis}\theta_1, \quad z_2 = r_2 \operatorname{cis}\theta_2$

$$\frac{z_1}{z_2} = \frac{r_1 \operatorname{cis}\theta_1}{r_2 \operatorname{cis}\theta_2}$$

$$= \frac{r_1(\cos\theta_2 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)} \times \frac{\cos\theta_2 - i\sin\theta_2}{\cos\theta_2 - i\sin\theta_2}$$

$$= \frac{r_1}{r_2}\frac{(\cos\theta_1\cos\theta_2 - i\cos\theta_1\sin\theta_2 + i\sin\theta_1\cos\theta_2 - i^2\sin\theta_1\sin\theta_2)}{(\cos\theta_2)^2 - (i\sin\theta_2)^2}$$

$$= \frac{r_1}{r_2}\frac{\left[(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 - \sin\theta_2\cos\theta_1)\right]}{\cos^2\theta_2 + \sin^2\theta_2}$$

$$\because \begin{cases} \cos\alpha \cdot \cos\beta + \sin\alpha\sin\beta = \cos(\alpha-\beta) \\ \sin\alpha\cos\beta - \sin\beta\cos\alpha = \sin(\alpha-\beta) \\ \cos^2\theta + \sin^2\theta = 1 \end{cases}$$

$= \frac{r_1}{r_2}\left[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)\right]$

Example:-

Express $\frac{1+\cos 2\theta - i\sin 2\theta}{\cos 2\theta + i\sin 2\theta}$ into $A+iB$.

Solve:-

$$\left\{\begin{aligned} z_1 &= r_1 \operatorname{cis}\theta_1 \\ z_2 &= r_2 \operatorname{cis}\theta_2 \\ z_1 z_2 &= r_1 r_2 \operatorname{cis}(\theta_1+\theta_2) \\ \frac{z_1}{z_2} &= \frac{r_1}{r_2}(\operatorname{cis}\theta_1-\theta_2) \\ z &= r(\cos\theta + i\sin\theta) \end{aligned}\right.$$

$$\frac{1+\cos 2\theta - i\sin 2\theta}{\cos 2\theta + i\sin 2\theta}$$

$$\because \left\{\begin{aligned} 1+\cos 2\theta &= 2\cos^2\theta \\ \sin 2\theta &= 2\sin\left(\tfrac{1}{2}\right)\cdot\cos\left(\tfrac{\theta}{2}\right) \end{aligned}\right.$$

$$= \frac{2\cos^2\theta - i(2\sin\theta\cdot\cos\theta)}{\cos 2\theta + i\sin 2\theta}$$

$$= \frac{2\cos\theta(\cos\theta - i\sin\theta)}{\cos 2\theta + i\sin 2\theta}$$

$$\left\{\begin{aligned} -\sin\theta &= \sin(-\theta) \\ \cos\theta &= \cos(-\theta) \end{aligned}\right.$$

$$= \frac{2\cos\theta\cdot(\cos(-\theta) + i\sin(-\theta))}{\cos 2\theta + i\sin 2\theta}$$

$$= 2\cos\theta\,[\cos(-\theta+2\theta) + i\sin(-\theta-2\theta)]$$

$$= 2\cos\theta\,[\cos(-3\theta) + i\sin(-3\theta)]$$

$$= 2\cos\theta\,(\cos 3\theta) - i\sin 3\theta$$

$$= \underbrace{2\cos\theta\cos 3\theta}_{A} + i\underbrace{(-2\cos\theta\sin 3\theta}_{B}$$

$$= A + iB$$

Lecture #031-

Conjugates Modulus and locus of a Complex Number.

* Conjugate of a Complex Number and its properties?

(i) $\bar{\bar{z}} = z$

(ii) $z = \bar{z} \iff$ '$z$' is pure real

(iii) $z = -\bar{z} \iff$ '$z$' is pure Imaginary

(iv) $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$

Conjugate of Sum = Sum ~~of~~ Conjugate

(v) $\overline{z_1 z_2} = \bar{z}_1 \cdot \bar{z}_2$

(vi) $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}$

(vii) $(\overline{z^n}) = (\bar{z})^n$

(viii) $\text{Re}\, z = \frac{z + \bar{z}}{2}$

(ix) $\text{Im}\, z = \frac{z - \bar{z}}{2i}$

(x) $z_1 \bar{z}_2 + \bar{z}_1 z_2 = 2\text{Re}(z_1 \bar{z}_2) = 2\text{Re}(\bar{z}_1 z_2)$

- **Modulus of a Complex Number and its properties :-**

$|z| = \sqrt{x^2+y^2}$ • order relation

(i) $|z| \geq 0$ (non-negative)

$x^2 \geq 0;\ y^2 \geq 0,\ \forall\, x, y \in \mathbb{R}$

$\Rightarrow x^2+y^2 \geq 0 \Rightarrow \sqrt{x^2+y^2} \geq 0$

(ii) $-|z| \leq \mathrm{Re}\, z \leq |z|$;

$\Rightarrow -\sqrt{x^2+y^2} \leq x \leq \sqrt{x+y^2} \Rightarrow -\sqrt{x+y^2} \leq y \leq \sqrt{x^2+y^2}$

(iii) $|z| = |\bar{z}| = |-z| = |-\bar{z}|$

(iv) $z\bar{z} = |z|^2$

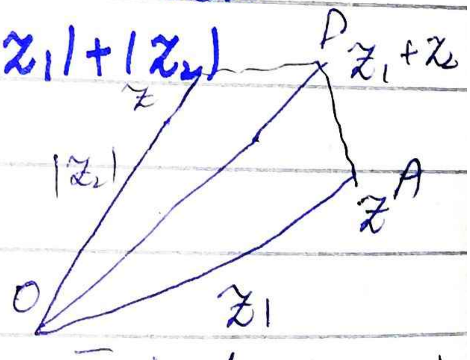
(v) $|z^n| = |z|^n$

(vi) $|z_1 z_2| = |z_1| \cdot |z_2|$

(vii) $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}$

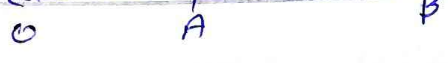
- **Triangular Inequality in Complex Numbers :-**

$||z_1| - |z_2|| \leq |z_1 + z_2| \leq |z_1| + |z_2|$




$|\overline{OP}| < |\overline{OA}| + |\overline{AP}|$

$\Rightarrow |z_1+z_2| < |z_1| + |z_2|$



$|OP| = |ON| + |NP|$

$|z_1 + z_2| = |z_1| + |z_2|$

**Proof:-**

for R.H.S

$|z|^2 = z \cdot \bar{z}$

$|z_1 + z_2|^2 = (z_1 + z_2)(\bar{z}_1 + \bar{z}_2) \quad \because \overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$

$= z_1 \bar{z}_1 + z_1 \bar{z}_2 + \bar{z}_1 z_2 + z_2 \cdot \bar{z}_2$

$\because z\bar{z} = |z|^2$

$= |z_1|^2 + |z_2|^2 + z_1 \bar{z}_2 + \bar{z}_1 z_2$

$= |z_1|^2 + |z_2|^2 + 2Re(z_1 \bar{z}_2)$

$|z_1 + z_2|^2 \leq |z_1|^2 + |z_2|^2 + 2|z_1||\bar{z}_2|$

$|z_1 + z_2|^2 \leq (|z_1| + |z_2|)^2 \ \forall\ x, y \in R, x, y \geq$

$|z_1 + z_2| \leq |z_1| + |z_2| \longrightarrow (A)$

It can be extended to

$|z_1 + z_2 + \cdots + z_n| \leq |z_1| + |z_n| + \cdots |z_n|$

$\left| \sum_{k=1}^{n} z_n \right| \leq \sum_{j=1}^{n} |z_j|$

for L.H.S & Middle

$|z_1| = |z_1 + z_2 - z_2| = |z_1 + z_2 + (-z$

$|z_1| \leq |z_1 + z_2| + |-z_2|$

$|z_1| \leq |z_1 + z_2| + |z_2|$

$|z_1| - |z_2| \leq |z_1 + z_2| \longrightarrow (B)$

Now for $z_2 = |z_1 + z_2 - z_1| = |z_1 + z_2 + (-$

$|z_2| \leq |z_1 + z_2| + |z_1|$

$-|z_1| + |z_2| \leq |z_1 + z_2|$

$-|z_1 + z_2| \leq (|z_1| - |z_2|) \longrightarrow (b)$

on Combining (a) and (b)

$\Rightarrow -|z_1 + z_2| \leq |z_1| - |z_2| \leq |z_1 + z_2|$

$\forall\ x, a \in \mathbb{R};$

if $-a \leq x \leq a \Rightarrow |x| \leq a$

$||z_1| - |z_2|| \leq |z_1 + z_2| \longrightarrow (B)$

on Combing (A) and (B) we get real inequality.

**Example:-**

find the Maximum and Minimum value of $f(x) = \left[\frac{2}{3+ie^{ix}}\right] \in \mathbb{R} \geq 0$

Triangular Inequality ; $|z_1| - |z_2| \leq |z_1 + z_2|$

$\leq |z_1| + |z_2| \longrightarrow (A)$

$\therefore |3 + ie^{ix}| \leq |3| + |ie^{ix}| = 3 + 1 = 4$

$|3 + ie^{ix}| \leq 4$

$\frac{1}{3 + ie^{ix}} \geq 4$

$\frac{2}{3+ie^{ix}} > \frac{2}{4}$

$\frac{|2|}{|3+ie^{ix}|} \geq \frac{1}{2}$

$$f(x) = \left| \frac{2}{3+ie^{ix}} \right| \geq \frac{1}{2}$$

$$|z_1| - |z_2| \leq |z_1 + z_2| \geq |z_1|$$

Now for $|3+ie^{ix}| \geq |3| - |ie^{ix}|$

$$= 3 - 1 = 2$$

$$\frac{1}{3+ie^{ix}} \leq \frac{1}{2}$$

Minium Value is $\frac{1}{2}$

Max value is 1

# Practice Question of lecture 01 to 03

## Question # 01:-

Show that $i^{2n-1} + i^{2n} + i^{2n+1} + i^{2n+2} = 1 - i$ where $n \in E$ = set of positive even integers.

## Solution:-

$$\Rightarrow i^{2n-1} + i^{2n} + i^{2n+1} + i^{2n+2}$$

$$= i^{2n}\, i^{-1} + i^{2n} + i^{2n}\, i^{+1} + i^{2n}\, i^{2}$$

$$= [i^2]^n \left[\frac{1}{i}\right] + (i^2)^n + [i^2]^n i + [i^2]^n (-1) \quad \because i^2 = -1$$

$$= (-1)^n \frac{1}{i \cdot i} + (-1)^n + (-1)^n i + (-1)^n (-1)$$

$$= (1)(-i) + (1) + (1)i + (1)(-1) \quad \because n \in E \Rightarrow (-1)^n = 1$$

$$= -i + 1 + i - 1 = 0$$

## Question #02:-

Evalute $\frac{3}{i} - \frac{i}{3}$

## Solution:-

$$\frac{3}{i} - \frac{i}{3} = \frac{3}{i} \times \frac{i}{i} - \frac{i}{3}$$

$$= \frac{3i}{i^2} - \frac{i}{3} = \frac{3i}{(-1)} - \frac{i}{3} = -3i - \frac{i}{3} = \left(-3 - \frac{1}{3}\right)i$$

$$= \left(-\frac{10}{3}\right)i$$

## Question #03:-

Discuss how many anti-clock quarter rotations, the following complex number will have? $-5i$, $-6$ and $8i$.

## Solution :-

$i$ graphically represents an anticlock quarter rotation of $\frac{\pi}{2}$ radians.

$\Rightarrow$

(i) $-5i$ represents one clock ~~wise~~ quarter rotation.

(ii) $-6 = 6(-1) = 6i^2 = 6i \cdot i$ . represent two anti-clock ~~wise~~ rotations.

(iii) $8i$ represents one anti-clock quarter rotations.

## Question 04:-

Simplify :- $i^{11} + i^{40} + i^{30}$.

$$i^{11} + i^{40} + i^{30} = (i^2)^5 \cdot i + (i^2)^{20} + (i^2)^{15}$$

$$= (-1)^5 i + (-1)^{20} + (-1)^{15}$$

$$= -i + 1 - 1 = -i$$

## Question # 05:-

Discuss why $i \neq 0$.

let $i = 0$ xing both side by $i$, we get,

$$i \cdot i = 0 \cdot i$$
$$i^2 = 0$$
$$-1 = 0$$

which is false. Hence $i \neq 0$

## Question #06:-

find Principal and all other possible arguments of $1+i$.

**Solution:-**

Here given that $z = x+iy = 1+i$

So $x = 1 > 0$ and $y = 1 > 0$

Therefore, terminal ray of Arg z lies in $1^{st}$ quadrant.

$\therefore Arg(z) = \tan^{-1}\left[\frac{y}{x}\right] = \tan^{-1}\left[\frac{1}{1}\right] = \tan^{-1}(1)$
$= \pi/4$

And $arg(z) = \frac{\pi}{4} + 2n\pi, n \in z$, that gives the all possible arguments.

## Question #07:-

Express $-1-\sqrt{3}i$ into polar form.

$z = x+iy = -1-\sqrt{3}i$

let :-

$\Rightarrow |z| = \sqrt{1+3} = 2$

So, $\theta = \tan^{-1}(\sqrt{3}) = \frac{\pi}{3}$

Since $-1-\sqrt{3}i$ lies on $3^{rd}$ quardant, so

$\theta = \frac{\pi}{3} - \pi = -\frac{2\pi}{3}$

Polar form is : $2C$ is $\left(-\frac{2\pi}{3}\right)$

Question # 08:-

Solve $x = \sqrt{-3+\sqrt{-3\sqrt{-3+ - \infty}}}$ and show that the difference of its roots is pure imaginary.

Solution.

$x = \sqrt{-3+\sqrt{-3\sqrt{-3+ --- \infty}}}$

Squaring:-

$x^2 = -3 + \underbrace{\sqrt{-3+\sqrt{-3+ -- \infty}}}_{x}$

$x^2 = -3 + x \Rightarrow$

$x^2 - x + 3 = 0$

$x = \frac{-(-1) \pm \sqrt{1-4(1)(3)}}{2}$

$x = \frac{1 \pm \sqrt{-12}}{2} = \frac{1 \pm i\sqrt{11}}{2}$ are the roots of given equation

Hence roots are: $x_1 = \frac{1+i\sqrt{11}}{2}$ and $x_2 = \frac{1-i\sqrt{11}}{2}$

Now difference is $= x_1 - x_2 = \left(\frac{1+i\sqrt{11}}{2}\right) - \left(\frac{1-i\sqrt{11}}{2}\right)$

$= \frac{\cancel{1}+i\sqrt{11} - \cancel{1} + i\sqrt{11}}{2} = \frac{\cancel{2}i\sqrt{11}}{\cancel{2}} = i\sqrt{11}$ i.e

Pure imaginary:-

## Question # 09:-

for a Complex number $z \in \mathbb{C}$ if, $|z| = 3$ and $Arg(z) = \frac{\pi}{3}$, then find $\frac{1}{\bar{z}}$.

$$\frac{1}{\bar{z}} = \frac{z}{\bar{z}z} = \frac{z}{|z|^2} \quad \because \bar{z} = \bar{z}z$$

$$= \frac{1}{(3)^2} z = \frac{1}{9}(|z| cis\theta) = \frac{1}{9}\left(3\left(\cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right)\right)\right)$$

$$= \frac{1}{3}\left(\frac{1}{2} + \frac{i\sqrt{3}}{2}\right) = \frac{1}{6}\left(1 + \sqrt{3}i\right)$$

## Question # 10:-

let $|z_1| = 2$ and $z_2 = -1+\sqrt{3}i$, Then by using Triangular Inequality. Find the extreme value of $\left|\frac{z_1+z_2}{2}\right|$.

Given that $z_1 = 2$, $z_2 = -1+\sqrt{3}i$

$$|z_1| = \sqrt{1+3} = 2$$

Now by using triangular Inequality:-

$$||z_1| - |z_2|| \leq |z_1 + z_2| \leq |z_1| + |z_2|$$

$$|1 - 2| \leq |z_1 + z_2| \leq 1 + 2$$

$$1 \leq |z_1 + z_2| \leq 3$$

$$\frac{1}{2} \leq \left|\frac{z_1 + z_2}{2}\right| \leq \frac{3}{2}$$

$$\text{Max}\left\{\left|\frac{z_1+z_2}{2}\right|\right\} = \frac{3}{2}$$

$$\text{Min}\left\{\left|\frac{z_1+z_2}{2}\right|\right\} = \frac{1}{2}.$$

## Question #11

Show that the locus of point $P(z)$ stisfying : $\left|\frac{1+iz}{\bar{z}+1}\right| = 2$ is

$$3x^2+3y^2+8x-2y+3=0$$

**Solution :-**

$$\left|\frac{1+i\bar{z}}{\bar{z}+1}\right| = 1$$

$$\left|\frac{1+i\,\overline{(x+iy)}}{\overline{(x+iy)}+1}\right| = 1$$

$$\frac{|1+i\,\overline{(x+iy)}|}{|\overline{(x+iy)}+1|} = 2 \quad \because \left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$$

$$|1+i(x-iy)| = 2|(x-iy)+1|$$

$$|(1+y)+ix| = 2|(x+1)-iy|$$

$$\left(\sqrt{(1+y)^2+x^2}\right)^2 = \left(2\sqrt{(x+1)^2+(-y)^2}\right)^2$$

$$(1+y)^2+x^2 = 4((x+1)^2+(-y)^2)$$

$$1+2y+y^2+x^2 = 4(x+1)^2+(-y)^2$$

$$1+2y+y^2+x^2 = 4(x^2+2x+1+y^2)$$

$$1+2y+y^2+x^2=4x^2+8x+4+4y^2$$
$$4x^2+8x+4+4y^2-1-2y-y^2-x^2=0$$
$$3x^2+3y^2+8x-2y+3=0$$ **Answer**

## lecture # 04:-

## Euler's formula, De Moivre's theorem and their applications

### 1. Euler's formula:-

$$\cos x + i\sin x = e^{ix}$$

let $z = x+iy = (r\cos x + i\sin x) = r\,\mathrm{cis}\,x \rightarrow (1)$

By Maclaurin Series:-

$$\cos x = 1-\frac{x^2}{2}+\frac{x^4}{4!}-\text{---}$$

$$\sin x = x-\frac{x^3}{3!}+\frac{x^5}{5!}+\text{---}$$

$$i\sin x = ix-\frac{ix^3}{3!}+\frac{ix^5}{5!}$$

$$= ix+i.i^2\frac{x^3}{3!}+i\,i^4\frac{x^5}{5!}+\text{---}$$

$$i\sin x = ix+\frac{(ix)^3}{3!}+\frac{(ix^5)}{5!}+\text{---} \rightarrow (ii)$$

$$\cos x = 1-\frac{x^2}{2}+\frac{x^4}{4!}+\text{---}$$

$$\cos x = 1+\frac{(ix)^2}{2!}+\frac{(ix)^4}{4!} \longrightarrow (iii)$$

Put in eq ①

$$Z = r\left[\left\{1 + \frac{(ix)^2}{2!} + \frac{(ix)^4}{4!} + \cdots\right\} + \left\{ix + \frac{(ix)^3}{3!} + \frac{(ix)^5}{5!} + \cdots\right\}\right]$$

$$Z = r\left[1 + (ix) + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \frac{(ix)^5}{5!} + \cdots\right]$$

$$\because \left\{ e^{\alpha} = 1 + \alpha + \frac{\alpha^2}{2!} + \frac{\alpha^3}{3!} + \frac{\alpha^4}{4!} + \cdots \right.$$

$$Z = re^{ix}$$

$$\cancel{r}(\cos x + i\sin x) = \cancel{r}e^{ix}$$

$$\cos x + i\sin x = e^{ix}$$

Put $x = \pi$

$$\cos\pi + i\sin\pi = e^{i\pi}$$

$$-1 + 0 = e^{i\pi}$$

$$\boxed{e^{i\pi} + 1 = 0} \Rightarrow \text{Euler's Identity}$$

**Examples of Euler's formula:-**

If $Z_k = \cos\left(\frac{\pi}{2k}\right) + i\sin\left(\frac{\pi}{2k}\right)$; Then

evalute $\prod_{k=1}^{\infty} Z_k$.

$\prod$ = product

$$Z_k = \cos\left(\frac{\pi}{2k}\right) + i\sin\left(\frac{\pi}{2k}\right) = e^{i\left(\frac{\pi}{2k}\right)}$$

$$\prod_{K=1}^{\infty} = z_1 \cdot z_2 \cdot z_3 + \cdots \infty = e^{i\left(\frac{\pi}{2}\right)} \cdot e^{i\left(\frac{\pi}{2^2}\right)} \cdot e^{i\left(\frac{\pi}{2^3}\right)} \cdots$$

$$= e^{i\left(\frac{\pi}{2}\right) + i\left(\frac{\pi}{2^2}\right) + i\left(\frac{\pi}{2^3}\right) + \cdots \infty}$$

$$e^{i\frac{\pi}{2}\left[1 + \frac{1}{2} + \frac{1}{2^2} + \cdots \infty\right]}$$

$\Rightarrow$ { Here $1 + \frac{1}{2} + \frac{1}{2^2} + \cdots$ is a geometric Series.

$\Rightarrow a_1 = a = 1; \quad r = \frac{a_2}{a_1} = \frac{1/2}{1} = 1/2$

$$\left\{ S_\infty = \frac{a}{1-r} = \frac{1}{1-1/2} = \frac{1}{1/2} = 2 \right.$$

$$= e^{i\frac{\pi}{\cancel{2}}(\cancel{2})}$$

$\Rightarrow e^{i\pi} = \cos\pi + i\sin\pi$

$-1 + i(0) = -1$

$e^{i\pi} = -1$ Euler's Identity

## * DeMoivre's Theorem (1- Integers Case)

for any $z = x + iy = r(\cos\theta + i\sin\theta) = re^{i\theta}$

$r(\cos\theta + i\sin\theta) = re^{i\theta}$

$(\cos\theta + i\sin\theta)^n = (e^{i\theta})^n$ for any $n \in \mathbb{Z}^+$

$= e^{in\theta}$

$\boxed{(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta}$ DeMoivre's Theorem

for $n \in \mathbb{Z}^-$

$(\cos\theta + i\sin\theta)^{-m} = (e^{i\theta})^{-m}$

$$e^{-im\theta} = \cos(-m\theta) + i\sin\theta(-m\theta)$$
$$= \cos m\theta - i\sin m\theta \quad \because \cos(-\alpha) = \cos\alpha$$
$$(\cos\theta + i\sin\theta)^{-m} = \cos m\theta - i\sin m\theta \quad \sin(-\alpha) = \sin\alpha$$

## *De Moivre Theorem (Q-Rational Case)

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$
$$n = \frac{p}{q} \in Q \; ; \; Q \neq 0$$

for $\frac{1}{q} \in q$ :-

$$(\cos\theta + i\sin\theta)^{1/q} = (e^{i\theta})^{1/q} = e^{i\frac{\theta}{q}}$$

$$e^{i\theta/q} = \cos\frac{\theta}{q} + i\sin\frac{\theta}{q} \rightarrow (1)$$

For $\sin\theta = \sin\theta \pm 2k\pi$
$\cos\theta = \cos(\theta \pm 2k\pi)$

$$(\cos\theta + i\sin\theta)^{1/q} = \cos\left(\frac{\theta + 2\pi k}{q}\right) + i\sin\left(\frac{\theta + 2k\pi}{q}\right)$$

Put $k = 0$

$$(\cos\theta + i\sin\theta)^{1/q} = \cos\frac{\theta}{q} + i\sin\frac{\theta}{q} \rightarrow (B)$$

(B) is one of the value of (A), whenever $k=0$ for $\frac{1}{q} \in Q$

Wolfram Alpha

from (B)

$$\left[(\cos\theta + i\sin)^{1/q}\right]^p = \left[\cos\frac{\theta}{q} + i\sin\frac{\theta}{q}\right]^p$$

$$= \cos\frac{p}{q}\theta + i\sin\frac{p}{q}\theta$$

## • De Moivres Theorem Example.

Evalute $(-1+i)^{100}$

let $z = x+iy = -1+i$

$x = -1, \quad y = 1$

$r = \sqrt{(-1)^2+(1)^2}$ $\Rightarrow$ $\mathbb{R}(\theta)$ lies in 2nd quad.

$r = \sqrt{2}$

$$\operatorname{Arg} z = \theta = \pi - \tan^{-1}\left(\frac{y}{x}\right)$$

$$= \pi - \tan^{-1}\left(\frac{1}{1}\right)$$

$$= \pi - \tan^{-1}(1)$$

$$\theta = \pi - \frac{\pi}{4} \Rightarrow \frac{3\pi}{4}$$

Now $z = r(\cos\theta + i\sin\theta) = \sqrt{2}\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}$

$$(-1+i)^{100} = \left[\sqrt{2}\operatorname{cis}\left(\frac{3\pi}{4}\right)\right]^{100} = (\sqrt{2})^{100}\operatorname{cis}\left(\frac{3\pi}{4}\times 100\right)$$

$$= (2^{1/2})^{100}\operatorname{cis}(3\pi\times 25)$$

$$= 2^{50}\cdot\operatorname{cis}(74\pi+\pi)$$

$$= 2^{50}\cdot\operatorname{cis}(37(2\pi)+\pi)$$

$$= 2^{50}\cdot\operatorname{cis}(\pi)$$

$$= 2^{50}\cos\pi + i\sin\pi$$

$$= 2^{50}(-1) + i(0)$$

$$= -2^{50}$$

lecture #05:-

## DeMoivers Theorem (Application-1)

To express ~~Cos~~ Cos$n\theta$ and Sin$n\theta$ as finite sums of powers of Trigonometric function of $\theta$, where $n$ is positive:-

$(\cos\theta + i\sin\theta)^2 = \cos^2\theta + (i\sin\theta)^2 + 2\cos\theta\, i\sin\theta$

$= \cos^2\theta - \sin^2\theta + i(2\cos\theta\sin\theta)$

By De Moiver's law:-

$\cos 2\theta + i\sin 2\theta = \cos^2\theta - \sin\theta + i(2\cos\theta\sin\theta)$

Real:-

$\cos 2\theta = \cancel{\cos^2}\cos^2\theta - \sin^2\theta$

$i\sin 2\theta = 2\cos\theta\sin\theta$

### Bionomial cofficent:-

$(a+b)^2 = a^2 + 2ab + b^2 = \binom{2}{0}a^{2-0}\cdot b^0 + \binom{2}{1}a^{2-1}\cdot b + \binom{2}{2}a^{2-2}\cdot b^2$

$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

$(a+b)^n = \binom{n}{0}a^{n-0}b^0 + \binom{n}{1}a^{n-1}\cdot b^1 + \binom{n}{2}a^{n-2}\cdot b + \dots \binom{n}{r}a^{n-r}\cdot b^r + \binom{n}{r+1}a^{n-(r+1)}b^{r+1} + \dots + \binom{n}{n-1}a^{n-(n-1)}b^{n-1} + \binom{n}{n}a^{n-n}b^n$

# ② De'Moiver's Theorem (Application-1 examples)

find the trigonometric Identities for $\sin 5\theta$, $\cos 5\theta$, $\tan 5\theta$.

let $\sin\theta = s$, $\cos\theta = c$, $\tan\theta = t$

By De'Moiver's theorem:-

$$\cos 5\theta + i\sin 5\theta = (\cos\theta + i\sin\theta)^5$$

$$= (c + is)^5$$

$$= \binom{5}{0} c^{5-0} (is)^0 + \binom{5}{1} c^{5-1} (is)^1 + \binom{5}{2} c^{5-2} (is)^2$$

$$+ \binom{5}{3} c^{5-3} (is)^3 + \binom{5}{4} c^{5-4} (is)^4 + \binom{5}{5} c^{5-5} (is)^5$$

$$= \frac{5!}{(5-0)!\,0!} c^5 \cdot 1 + \frac{5!}{(5-1)!\,1!} c^5 (is) + \frac{5!}{(5-2)!\,2!} c^5 (-1) s^2$$

$$+ \frac{5!}{(5-3)!\,3!} c^2 (-1) s^3 + \frac{5!}{(5-4)!\,4!} c \cdot 1 \cdot s^4 + \frac{5!}{(5-5)!\,5!} \cdot 1 \cdot is^5$$

$$= c^5 + 5c^4 (is) + 10c^3(-s^2) + 10c^2(-is^3) + 5cs^4 + is^5$$

$$\cos 5\theta + i\sin 5\theta = (c^5 - 10c^3 s^2 + 5cs^4) + i(s^5 - 10c^2 s^3 + 5c^4 s)$$

equating Real and Imaginary parts.

$$\cos 5\theta = c^5 - 10c^3 s^2 + 5cs^4 \longrightarrow (i)$$

$$\sin 5\theta = s^5 - 10c^2 s^3 + 5c^4 s \longrightarrow (ii)$$

$\sin\theta = 2\sin\theta\cos\theta$

$\cos^2\theta + \sin\theta = 1$

$c^2 = 1 - s^2$

$s^2 = 1 - c^2$

$s^4 = (s^2)^2$

from equation 1

$\cos 5\theta = c^5 - 10c^3 s^2 + 5cs^4$

$= c^5 - 10c^3(1-c^2) + 5c(1-c^2)^2$

$= c^5 - 10c^3 + 10c^5 + 5c(1 + c^4 - 2c^2)$

$= c^5 - 10c^3 + 10c^5 + 5c + 5c^5 - 10c^3$

$= c^5 - 10c^3 + 10c^5 + 5c$

$\cos 5\theta = 16c^5 - 20c^3 + 5c$

$\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta.$

from equation 2

$\sin 5\theta = s^5 - 10c^2 s^3 + 5c^4 s$

$= s^5 - 10(1-s^2)s^3 + 5s(1-s^2)^2$

$= s^5 - 10s^3 + 10s^5 + 5s(1 + s^4 - 2s^2)$

$= s^5 - 10s^3 + 10s^5 + 5s + 5s^5 - 10s^2$

$= 16s^5 - 20s^3 + 5c$

$\sin 5\theta = 16\sin^5\theta - 20\sin^3\theta + 5\sin\theta.$

$$\tan 5\theta = \frac{\sin 5\theta}{\cos 5\theta}$$

$$\frac{\sin 5\theta}{\cos 5\theta} = \frac{s^5 - 10c^2s^3 + 5c^4s}{c^5 - 10c^3s^2 + 5c\cdot s^4}$$

Dividing Numetor and Denometar by $c^5$

$$\tan 5\theta = \frac{s^5/c^5 - \frac{10c^2s^3}{c^5} + 5c^4s/c^5}{\frac{c^5}{c^5} - \frac{10c^3s^2}{c^5} + \frac{5cs^4}{c^5}}$$

$$= \frac{t^5 - 10t^3 + 5t}{1 - 10t^2 + 5t^4}$$

$$\tan 5\theta = \frac{\tan^5\theta - 10\tan^3\theta + 5\tan\theta}{1 - 10\tan^2\theta + 5\tan^4\theta}$$

# De'Moivers Theorem Application '2'

Express $\cos^{-1}\theta$ and $\sin^{-1}\theta$ as Sum of Sines and Cosines of Mutiples of $\theta$

let

$x = \cos\theta + i\sin\theta \rightarrow$ (I)

$$\frac{1}{x} = \frac{1}{\cos + i\sin\theta} = (\cos\theta + i\sin\theta)^{-1}$$

$= +\cos(-\theta) + i\sin(-\theta)$

$= \cos\theta - i\sin\theta \rightarrow$ (2)

By (1) + (2)

$$x + \frac{1}{x} = (\cos\theta + i\sin\theta) + (\cos\theta - i\sin\theta)$$

$$\boxed{x + \frac{1}{x} = 2\cos\theta}$$

By (1) - (2)

$$x - \frac{1}{x} = \cos\theta + i\sin\theta - \cos\theta - i\sin\theta$$

$$\boxed{x - \frac{1}{x} = 2i\sin\theta}$$

for $n^{th}$ powers

$$x^n = (\cos\theta + i\sin\theta)^n$$

$$= \cos n\theta + i\sin n\theta \rightarrow (3)$$

$$\frac{1}{x^n} = (\cos\theta + i\sin\theta)^{-n}$$

$$= \cos n\theta - i\sin n\theta \rightarrow (4)$$

By (3) + (4)

$$\boxed{x^n + \frac{1}{x^n} = 2\cos n\theta}$$

By (3) - (4)

$$\boxed{x^n - \frac{1}{x^n} = 2i\sin n\theta}$$

Example :-

$\cos^4\theta \cdot \sin^3\theta$ as sum of sines and cosines of multiples of $\theta$

$$(2\cos\theta)^4(2i\sin\theta)^3 = \left(x+\frac{1}{x}\right)^4\left(x-\frac{1}{x}\right)^3$$

$$2^4 \cdot \cos^4\theta \cdot 2^3 i^3 \sin^3\theta = \left(x+\frac{1}{x}\right)\left(x+\frac{1}{x}\right)^3\left(x-\frac{1}{x}\right)^3$$

$$= \left(x+\frac{1}{x}\right)\left[\left(x+\frac{1}{x}\right)\left(x-\frac{1}{x}\right)\right]^3$$

$$\Rightarrow -2^7 i \cos^4\theta \sin^3\theta = \left(x+\frac{1}{x}\right)\left[x^2-\frac{1}{x^2}\right]^3$$

$\because (a-b)^3 = a^3 - b^3 - 3ab(a-b)$

$$= \left(x+\frac{1}{x}\right)\left[(x^2)^3 - \left(\frac{1}{x^2}\right)^3 - 3x^2\frac{1}{x^2}\left(x^2-\frac{1}{x^2}\right)\right]$$

$$= \left(x+\frac{1}{x}\right)\left[x^6 - \frac{1}{x^6} - 3x^2 + \frac{3}{x^2}\right]$$

$$= x^7 - \frac{1}{x^5} - 3x^3 + \frac{3}{x} + x^5 - \frac{1}{x^7} - 3x + \frac{3}{x^3}$$

$$= \left(x^7 - \frac{1}{x^7}\right) + \left(x^5 - \frac{1}{x^5}\right) - 3\left(x^3 - \frac{1}{x^3}\right) - 3\left(x - \frac{1}{x}\right)$$

$$= 2i\sin 7\theta + 2i\sin 5\theta - 3i\sin 3\theta - 3(2i\sin\theta)$$

$$-2^7 i\cos^4\theta\sin^3\theta = 2i\sin 7\theta + 2i\sin 5\theta - 3i\sin 3\theta - 3(2i\sin\theta)$$

By $\div$ing $i$ and on bothsides.

$$\therefore \cos^4\theta \cdot \sin^3\theta = \frac{-2}{2^7}\left[\sin 7\theta + \sin 5\theta - 3\sin 3\theta - 3\sin\theta\right]$$

$\cos^4\theta \sin^3\theta = \frac{1}{64}(\sin 7\theta + \sin 5\theta - 3\sin 3\theta - 3\sin\theta)$

Example:-

if $x+\frac{1}{x} = 2\cos\theta$, $y+\frac{1}{y} = 2\cos\phi$;

Then $x^m y^n + \frac{1}{x^m y^n} = 2\cos(m\theta + n\phi)$

$m, n \in \mathbb{Z}$.

$\because x+\frac{1}{x} = 2\cos\theta$

$$\frac{x^2+1}{x} = 2\cos\theta$$

$$x^2+1 = 2\cos\theta\, x$$

$$x^2 - 2\cos\theta\, x + 1 = 0$$

By Quardatic formulas-

$$x = \frac{-(-2\cos\theta) \pm \sqrt{(-2\cos\theta)^2 - 4(1)(1)}}{2(1)}$$

$$= \frac{2\cos\theta \pm \sqrt{4\cos^2\theta - 4}}{2}$$

$$= \frac{2\cos\theta \pm 2\sqrt{(-1)(1-\cos^2\theta)}}{2}$$

$$= \frac{\cancel{2}(\cos\theta \pm \sqrt{-\sin^2\theta})}{\cancel{2}}$$

$$= \cos\theta \pm \sqrt{-\sin^2\theta}$$

$$x = \cos\theta \pm i\sin\theta$$

So taking $x = \cos\theta + i\sin\theta$

$= \text{cis}\,\theta = e^{i\theta}$

$x^m = (e^{i\theta})^m = e^{im\theta} = \cos(m\theta + i\sin(m\theta))$

$= \text{cis}\, m\theta$

$x^{-m} = (e^{i\theta})^{-m} = e^{i(-m\theta)} = \cos(-m\theta) + i\sin(-m\theta)$

Similarly:-

$y^m = \text{cis}\, m\phi$ & $y^{-m} = \text{cis}(-m\phi)$

Now

$x^m y^n + \frac{1}{x^m y^n} = \text{cis}\, m\theta \cdot \text{cis}\, n\phi + \text{cis}(-m\theta) \cdot \text{cis}(-n\phi)$

$= \text{cis}(m\theta + n\phi) + \text{cis}(-m\theta - n\phi)$

$= [\cos(m\theta + n\theta) + i\sin(m\theta + n\theta)] + [\cos(-m\theta - n\theta) + i\sin(-m\theta - n\phi)]$

$= \cos(m\theta + n\phi) + i\sin\cancel{(m\theta + n\phi)}$

$+ \cos(m\theta + n\phi) - i\sin\cancel{(m\theta + n\phi)}$

$= 2\cos(m\theta - n\phi)$

# Lecture # 06:-

## Roots of a Complex Number

### 1. Roots of an equation (Introduction)

$f(x)=0$ , $y=f(x)=0$

Real root وہ ہوگی جو گراف x-axis پر meet کرے گی (نقطہ)

### (2) Square and cube root of unity.

let one root of unity

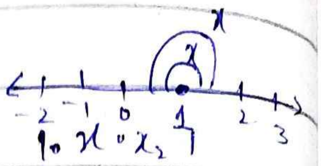
① $x=1$

② $x^2=1$

$x^2-1=0$

$(x-1)(x+1)=0$

$x=\pm 1$




③ cube roots:-

$x^3=1$

$\Rightarrow x^3-1=0$

$(x)^3-(1)^3=0$

$\because a^3-b^3=(a-b)(a^2+ab+b^2)$

$(x-1)(x^2+x+1)=0$

$x=1$ , $x^2+x+1=0$ i.e quad eq

$x=\dfrac{-1\pm\sqrt{1^2-4(1)(1)}}{2(1)}$

$x=\dfrac{-1\pm\sqrt{-3}}{2}$

$x_1=\dfrac{-1+\sqrt{-3}}{2}$ , $x_2=\dfrac{-1-\sqrt{-3}}{2}$

$$\omega = \frac{-1+\sqrt{3}i}{2} \quad ; \quad \omega^2 = \frac{-1-\sqrt{3}i}{2}$$

$$\omega^2 = \left(\frac{-1+\sqrt{-3}}{2}\right)^2$$

$$\omega^2 = \frac{(-1)^2 + (\sqrt{3})^2 + (2)(-1)(\sqrt{-3})}{2}$$

$$= \frac{-1-\sqrt{-3}}{2}$$

$1, \omega, \omega^2$ are cube roots of unity

**Observations :-**

(1) Sum ; $1+\omega+\omega^2 = 0$

(ii) product ; $1 \cdot \omega \cdot \omega^2 = \omega^3 = 1$

## 3. nth roots of unity :-

$x^n = 1$

$x^n - 1 = 0$

$\Rightarrow (x-1)(x^{n-1} + x^{n-2} + x^{n-3} + \cdots x^2 + x + 1) = 0$

$(x-1) = 0, \quad \boxed{x = 1}$

$(x^{n-1} + x^{n-2} + x^{n-3} + \cdots x^2 + x + 1) = 0$

$x^2 - 1 = 0$

$(x-1)(x+1) = 0$

$x^3 - 1 = 0$

$(x-1)(x^2 + x + 1) = 0$

$x^4 - 1 = 0$

$(x-1)(x^3 + x^2 + x + 1) = 0$

$x^n = 1 = 1 + i(0) = \cos(0) + i(\sin 0)$

$x^n = \text{cis}\,0$

$\sqrt[n]{x^n} = \sqrt[n]{\text{cis}\,0}$

$(x^n)^{1/n} = (\text{cis}\,0)^{1/n} = (\text{cis}(0+2k\pi))^{1/n}$

~~$x = \text{cis}\,\frac{1}{n}$~~

$$x = cis\left(\frac{2k\pi}{n}\right) \rightarrow (i) \quad k \in \mathbb{Z}$$

for $k = 0$;

$x_0 = cis(0) = \cos(0) + i\sin 0 = 1$

$k = 1$;

$$x_1 = cis\left(\frac{2(1)\pi}{n}\right) = cis\left(\frac{2\pi}{n}\right) = \omega$$

$k = 2$

$$x_2 = cis\left(\frac{2(2)\pi}{n}\right) = cis\left(\frac{4\pi}{n}\right) = \omega^2$$

$\vdots$

$k = n-1$

$$x_{n-1} = cis\left(\frac{2(n-1)\pi}{n}\right) = \omega^n$$

$k = n$,

$$x_n = cis\left(\frac{2n\pi}{n}\right)$$

$= cis 2\pi = \cos(2\pi) + i\sin(2\pi)$

$= 1 + i(0) = 1$

$1, \omega, \omega^2, —— \omega^{n-1}$ are $n^{th}$ roots of unity.

# 5. nth roots of any Real Number:-

$a \in \mathbb{R}$

$x^3 = a$

$x^3 - a = 0$

$x^3 - (a^{1/3})^3 = 0$

$x^3 - (a^{1/3})^3 = 0$

$\Rightarrow (x - a^{1/3})(x^2 + a^{1/3}x + a^{2/3}) = 0$

$x - a^{1/3} = 0 \;;\; x^2 + a^{1/3}x + a^{2/3} = 0$

$x = a^{1/3} \;,\; x = \dfrac{-a^{1/3} \pm \sqrt{a^{2/3} - 4(a^{2/3})}}{2 \cdot 1}$

$x = \dfrac{-a^{1/3} \pm \sqrt{(a^{1/3})^2 (1-4)}}{2}$

$= \dfrac{-a^{1/3} \pm a^{1/3}\sqrt{-3}}{2}$

$= a^{1/3}\left[\dfrac{1 \pm \sqrt{-3}}{2}\right]$

$= a^{1/3}(\omega, \omega^2)$

$x = a^{1/3}, a^{1/3}\omega, a^{1/3}\omega^2$

for nth roots of 'b'

$1, \omega, \omega^2 \text{ --- } \omega^{n-1}$

for $b \in \mathbb{R}$

$b^{1/n}, b^{1/n}\omega, b^{1/n}\omega^2 \text{ --- } b^{1/n}\omega^{n-1}$

## 6. Roots of any Complex number (Example):-

Find the 4th roots of $-2\sqrt{3}+2i$.

let $w^4 = -2\sqrt{3}+2i = x+iy$ ---(i)

$x = -2\sqrt{3}$, $y = 2$

T.ray of ~~ang~~ argu $= \theta$ lies in 2nd quad

$r = \sqrt{x^2+y^2}$

$r = \sqrt{(-2\sqrt{3})^2 + 2^2}$

$= \sqrt{16} = 4$

Argument of $z$ -

$\operatorname{Arg} z = \pi - \operatorname{Tan}^{-1}\left(\frac{y}{x}\right)$

$= \pi - \tan^{-1}\left(\frac{2}{\cancel{+2}\sqrt{3}}\right)$

$= \pi - \operatorname{Tan}^{-1}\left(\frac{1}{\sqrt{3}}\right)$

$= \pi - \pi/6$

$= 5\pi/6$

$w^4 = x+iy = r\operatorname{cis}\theta = 4\operatorname{cis}\theta$

$4\operatorname{cis}\left(\frac{5\pi}{6}\right)$

$\Rightarrow w^4 = 4\operatorname{cis}\left(\frac{5\pi}{6} + 2k\pi\right)$ ; $k \in \mathbb{Z}$

$= 4\operatorname{cis}\left(\frac{5\pi + 12k\pi}{6}\right)$

$$w^{4 \cdot \frac{1}{4}} = 4\,\mathrm{cis}\left\{\left(\frac{(5+12k)\pi}{6}\right)\right\}^{1/4}$$

$$w = 4^{1/4} \cdot \left[\mathrm{cis}\left(\frac{(5+12k)\pi}{6}\right)\right]^{1/4}$$

$$w = \sqrt{2} \cdot \mathrm{cis}\,\frac{1}{4}\left(\frac{(5+12k)\pi}{6}\right)$$

$$= \sqrt{2}\,\mathrm{cis}\left(\frac{(5+12k)\pi}{24}\right) \longrightarrow (1)$$

for $k = 0$

$$w_0 = \sqrt{2}\,\mathrm{cis}\left(\frac{5+12(0)\pi}{24}\right)$$

$$= \sqrt{2}\,\mathrm{cis}\,\frac{5\pi}{24}$$

for $k = 1$

$$w_1 = \sqrt{2}\,\mathrm{cis}\left(\frac{5\pi + 24\pi}{24}\right)$$

$$= \sqrt{2}\,\mathrm{cis}\left(\frac{29\pi}{24}\right)$$

for $k = 2$;

$$w_2 = \sqrt{2}\,\mathrm{cis}\left(\frac{5\pi + 36\pi}{24}\right)$$

$$= \sqrt{2}\,\mathrm{cis}\left(\left(\frac{41\pi}{24}\right) - 2\pi\right) = \sqrt{2}\,\mathrm{cis}\left(\frac{-7\pi}{24}\right)$$

$k = 3$;

$$w_3 = \sqrt{3}\,\mathrm{cis}\left(\frac{53\pi}{24}\right)$$

# Practice Question
## lecture # 04 to 06.

By using Polar and Euler's forms.-

**Q1:-**

Find all possible values of $(\cos\alpha + i\sin\alpha)^{1/5}$.

$(\cos\alpha + i\sin\alpha)^{1/5} = (\cos(\alpha + 2k\pi) + \sin(\alpha + 2k\pi))^{1/5}, \ k \in \mathbb{Z}$

$\therefore$ By De Moivre's Theorem:-

$(\cos\alpha + i\sin\alpha)^{1/5} = \cos\left[\frac{\alpha + 2k\pi}{5}\right] + i\sin\left(\frac{\alpha + 2k\pi}{5}\right)$

gives the required all possible values of $(\cos\alpha + i\sin\alpha)^{1/5}$.

**Q2:-**

Show that $e^{i\frac{\pi}{12}} e^{i\pi/4} e^{i2\pi/3} = -1$.

$e^{i\pi/12} e^{i\pi/4} e^{i2\pi/3} = e^{i\pi/12 + i\pi/4 + i2\pi/3}$

$= e^{i(\pi/12 + \pi/4 + 2\pi/3)}$

$= e^{i\left(\frac{\pi + 3\pi + 8\pi}{12}\right)}$

$= e^{i(12\pi/12)}$

$= e^{i\pi}$

| | |
|---|---|
| 2 | 12, 4, 3 |
| 3 | 6, 2, 3 |
| 2 | 2, 2, 1 |
| | 1, 1, 1 |

= 12

By Euler's formula; $e^{i\pi}$

$= \cos\pi + i\sin(\pi)$

$= -1 + i(0) = -1$

$e^{i\pi/12} e^{i\pi/4} e^{i2\pi/3} = -1$ Hence proved.

Q3:-

Show that $\lim_{n\to\infty}(z_1 \cdot z_2 \cdots z_n) = -1$

Where $z_k = e^{i\pi/2^k}$, $k \in \mathbb{Z}$.

$$= \lim_{n\to\infty}(z_1 \cdot z_2 - - z_n)$$

$$= \lim_{n\to\infty}\left\{e^{\left(i\frac{\pi}{2^1} + \frac{\pi}{2^2} + - - + \frac{\pi}{2^n}\right)}\right.$$

$$= \lim_{n\to\infty} e^{i\pi\left(\frac{1}{2^1} + \frac{1}{2^2} + - - \frac{1}{2^n}\right)}$$

a geometric series

$$\therefore \left\{\begin{array}{l}\lim_{n\to\infty}\left(\frac{1}{2^1} + \frac{1}{2^2} + - - + \frac{1}{2^n}\right) = \frac{a}{1-r} \\ \text{where } a = r = 1/2 \\ \quad = 1/2 / 1 - 1/2 = 1\end{array}\right.$$

$$\therefore \lim_{n\to\infty}(z_1 \cdot z_2 - - z_n) = e^{i\pi}$$

$$= \cos(\pi) + i\sin(\pi) = -1$$

Question # 04:-

Prove that $(x+iy)^{\frac{s}{t}} + (x-iy)^{\frac{s}{t}} = 2(x^2+y^2)^{\frac{s}{2t}}$

$\cos\left(\frac{s}{t}\tan^{-1}\frac{y}{x}\right)$

$\because \forall (x+iy) \in \mathbb{C}$, the polar form is given by; $x+iy = re^{i\theta}$, where

$r = (x^2+y^2)^{1/2}$ and $\theta = \tan^{-1}\left(\frac{y}{x}\right)$

$\therefore (x+iy)^{\frac{s}{t}} + (x-iy)^{s/t}$

$= (re^{i\theta})^{\frac{s}{t}} - (re^{-i\theta})^{\frac{s}{t}}$

$= r^{\frac{s}{t}}\left(e^{i\theta\frac{s}{t}} - e^{-i\theta\frac{s}{t}}\right)$

$= r^{\frac{s}{t}}\left(\cos\left(\theta\frac{s}{t}\right) + i\sin\left(\theta\frac{s}{t}\right) + \cos\left(-\theta\frac{s}{t}\right) + i\sin\left(-\theta\frac{s}{t}\right)\right)$ $\because$ by Euler's

$= r^{p/q}\left(\cos\left(\theta\frac{s}{t}\right) + i\sin\left(\theta\frac{s}{t}\right) + \cos\left(\theta\frac{s}{t}\right) - i\sin\left(\theta\frac{s}{t}\right)\right)$

$\because$ $\cos(-\alpha) = \cos\alpha$ and $\sin(-\alpha) = -\sin\alpha$

$= 2r^{p/q}\cos\left(\theta\frac{s}{t}\right)$

$= 2\left\{(x^2+y^2)^{1/2}\right\}^{s/t}\cos\left(\frac{s}{t}\tan^{-1}\left(\frac{y}{x}\right)\right)$

$$= 2(x^2+y^2)^{\frac{1}{2i}} \cos\left(\frac{1}{i}\tan^{-1}\left(\frac{y}{x}\right)\right)$$

## Question #05:-

By DeMoivre's theorem, $(\cos x + i\sin x)^2 = \cos 2x + i\sin 2x$. then show that $\cos 2x = \cos^2 x - \sin^2 x$.

$$(\cos x + i\sin x)^2 = \cos 2x + i\sin 2x$$

$$(\cos x)^2 + (i\sin x)^2 + 2\cos x(i\sin x) = \cos 2x + i\sin 2x$$

$$\cos^2 x + i^2\sin^2 x + i(2\sin x\cos x) = \cos 2x + i\sin 2x$$

Comparing real parts;

$$\cos 2x = \cos^2 x - \sin^2 x$$

## Question #06:-

If $x - \frac{1}{x} = 2i\cos y$, then show that $x = \pm\sin y + i\cos y$.

we have given;

$$x - \frac{1}{x} = 2i\cos y$$

$$\frac{x^2-1}{x} = 2i\cos y$$

$$x^2 - 1 = (2i\cos y)x$$

$$x^2 - (2i\cos y)x - 1 = 0$$

which is quadratic in $x$;

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = (2i\cos y) \pm \sqrt{(2i\cos y)^2 - 4(1)(-1)} \Big/ 2$$

$$= (2i\cos y) \pm \sqrt{-4\cos^2 y + 4} \Big/ 2$$

$$= (2i\cos y) \pm \cancel{2}\sqrt{-\cos^2 y + 1} \Big/ \cancel{2}$$

$$= i\cos y \pm \sqrt{-\cos^2 y + 1}$$

$\because \cos^2 y = -\sin^2 y + 1$

$$\Rightarrow i\cos y \pm \sqrt{\sin^2 y}$$

$$x = i\cos y \pm \sin y$$

Hence proved.

**Question # 07:-**

**Evalute $(1+i)^{200}$.**

$1 + i = 1 + i(1)$

$x = y = 1 > 0 \Rightarrow$ Terminal ray of $\text{Arg}(z)$ lies in 1st quadrant.

$r = \sqrt{x^2 + y^2}$

$r = \sqrt{1^2 + 1^2} = \sqrt{2}$

$\text{Arg}(z) = \tan^{-1}\left(\frac{1}{1}\right) = \tan^{-1}(1) = \frac{\pi}{4}$

∴ Polar form :-

$1+i = r\,cis\,\theta = \sqrt{2}\,cis(\pi/4)$

$= \left(\sqrt{2}\,cis\left(\frac{\pi}{4}\right)\right)^{200}$

$= (\sqrt{2})^{200}\left(cis\left(\frac{\pi}{4}\right)\right)^{200}$

$= \left(2^{1/2}\right)^{200} cis\left(200\,\pi/4\right)$

∴ By De Moivre's theorem

$= 2^{100}\,cis(50\pi) = 2^{100}\,cis(49\pi+\pi)$

$= 2^{100}\,cis(\pi)$ ∵ $\theta' = \theta + 2k\pi,\ k \in \mathbb{Z}$

$= 2^{100}(\cos\pi + i\sin\pi)$

$= 2^{100}(-1+i(0))$

$= -2^{100}$

## Question # 08:-

Show that the sum of roots of equation : $x^2-4=0$ vanishes.

$x^2-4=0$

$(x^2)^2-(2)^2=0$

$(x^2-2)(x^2+2)=0$

$x^2-2=0,\quad x^2+2=0$

$x = \pm\sqrt{2},\quad \pm i\sqrt{2}$

Sum of roots $= (\sqrt{2})+(-\sqrt{2})+(i\sqrt{2})+(-i\sqrt{2})=$

Question #09:-

find the four roots of 'i'.

$$x^4 = i = 0 + i(1)$$

$$= \cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{2}\right) = cis\left(\frac{\pi}{2}\right)$$

$$= cis\left(\frac{\pi}{2} + 2k\pi\right) = cis\left((4k+1)\frac{\pi}{2}\right)$$

$$X_k = \left\{ cis\left((4k+1)\frac{\pi}{2}\right) \right\}^{1/4}$$

$\because$ By DMoivre's

$$= cis\left((4k+1)\frac{\pi}{8}\right)$$

Put $k = 0, 1, 2, 3$

$$x_0 = cis\left((4(0)+1)\frac{\pi}{8}\right) = cis\left(\frac{\pi}{8}\right)$$

$$x_1 = cis\left((4(1)+1)\frac{\pi}{8}\right) = cis\left(\frac{5\pi}{8}\right)$$

$$x_2 = cis\left((4(2)+1)\frac{\pi}{8}\right) = cis\left(\frac{9\pi}{8}\right)$$

$$x_3 = cis\left((4(3)+1)\frac{\pi}{8}\right) = cis\left(\frac{13\pi}{8}\right)$$

# lecture #07:-

## Basic Elementry Functions:-

1. Agebric function => polynomial

2. Exponeatial and lograthmic function

3. Trigonometic function and their inverse.

## Exponential function:-

for any $y \in \mathbb{R}$; Euler's formula,

$$e^{iy} = \cos y + i\sin y \rightarrow (1)$$

if $z = (x+iy) \in \mathbb{C}$

from (1)

xing both sides by "$e^x$"

$$e^x \cdot e^{iy} = e^x(\cos y + i\sin y)$$

$$e^{x+iy} = e^x(\cos y + i\sin y)$$

$$\boxed{e^z = e^x(\cos y + i\sin y)} \rightarrow (A)$$

i.e defined to be an exponeatial function for $z \in \mathbb{C}$.

Case (I)

$z$ is pure real so

$z_{Img} = 0 = y$

(A) $\Rightarrow e^z = e^x(\cos 0 + i\sin 0)$
$= e^x(1 + i0)$
$e^z = e^x$

Case (ii)

$z$ is pure imaginary $\Rightarrow \mathrm{Re}\, z = 0 = x$

(A) $\Rightarrow e^z = e^0(\cos y + i\sin y) = 1 \cdot e^{iy}$
$e^z = e^{iy}$

Example:-

for any $z = (x+iy) \in \mathbb{C}$ ; $e^z = 0$??

If $e^z = 0$, $e^x(\cos y + i\sin y) = 0$
$\Rightarrow \cos y + i\sin y = 0$ , $e^x \neq 0 \ \forall x \in \mathbb{R}$
$\cos y + i\sin y = 0 + i0$
Comparing real and Img
$\cos y = 0$, $\sin y = 0$ $y \in \mathbb{R}$.
There does not exist $y \in \mathbb{R}$ so that above holds.
$(\cos y + i\sin y) \neq 0 \Rightarrow e^x(\cos y + i\sin y)$
$e^z \neq 0$


Period of $(z) e^z$

Since $e^z = e^x(\cos y + i\sin y)$
for any $z = (x+iy) \in \mathbb{C}$


$e^z = e^x\{\cos(y + 2k\pi) + i\sin(y + 2k\pi)\}$
$= e^x e^{i(y + 2k\pi)}$
$= e^{x + iy + 2k\pi}$

$= e^{x+iy+i2k\pi}$

$e^{z} = e^{z+2k\pi i}$

$e^{z}$ is periodic with period $2k\pi i$
$k \in \mathbb{Z}$.

## Properties of complex Number:

i) $e^{z_1+z_2} = e^{z_1} e^{z_2}$

ii) $\dfrac{e^{z_1}}{e^{z_2}} = e^{z_1-z_2}$

(iii) $(e^{z})^{n} = e^{nz}$ ; $n \in \mathbb{Z}$.

$\Downarrow$

Taking $(e^{z})^{n}$
for any $z = (x+iy) \in \mathbb{C}$ ; $n \in \mathbb{Z}$

$(e^{z})^{n} = (e^{x+iy})^{n} = (e^{x} \cdot e^{iy})^{n}$

$= [e^{x}(\cos y + i\sin y)]^{n}$

$= (e^{x})^{n} \cdot (\cos y + i\sin y)^{n}$

$= e^{xn}(\cos ny + i\sin ny)$

$= e^{nx} \cdot e^{iny} = e^{nx+iny}$

$= e^{n(x+iy)} = e^{nz}$ R.H.S

Solve the equation :-

$$e^{z} = -1 \quad ; z \in \mathbb{C}$$

Solution :-

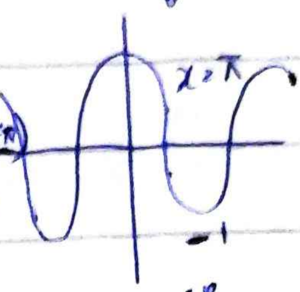
$$e^{z} = -1 = -1 + i(0)$$

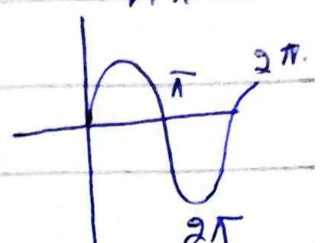
$$e^{z} = \cos\pi + i\sin\pi$$

$$e^{z} = \cos(\pi + 2k\pi) + i\sin(\pi + 2k\pi)$$

$$e^{z} = \cos\{(2k+1)\pi\} + i\sin\{(2k+1)\pi\}$$

$$e^{z} = e^{(2k+1)\pi i}$$




$$\begin{cases} e^{z_1} = e^{z_2} \quad \nRightarrow \quad z_1 = z_2 \\ \text{But} \quad z_1 = z_2 + 2k(i\pi) \end{cases}$$

$$z_1 = (2k+1)\pi i + 2k'(i\pi)$$

$$= 2k\pi i + \pi i + 2k i\pi$$

$$= (2k + 1 + 2k')\, i\pi$$

$$= [2(k + k') + 1]$$

$$= i(2(k'') + 1)$$

$$= i\,(2(k'' + 1))\,\pi$$

where $k + k' = k''$.

$k'' \in \mathbb{Z}$

Prove that $|e^{-2z}| < 1 \iff Re(z) > 0$; $z \in \mathbb{C}$.

let:-

$p: |e^{-2z}| < 1,$

& $q: Re\, z > 0, z \in \mathbb{C}$

$p \to q$ :-

$p$ is given & to prove $q$

$p \Rightarrow |e^{-2z}| < 1$ if $z = (x+iy) \in \mathbb{C}$

$\Rightarrow |e^{-2(x+iy)}| < 1$

$\Rightarrow |e^{-2x - 2iy}| < 1$

$\Rightarrow |e^{-2x} \cdot e^{-2iy}| < 1$

$\Rightarrow |e^{-2x}| \cdot |e^{-2iy}| < 1$

$\Rightarrow |e^{-2x}| \cdot |e^{i(-2y)}| < 1$

$\Rightarrow |e^{-2x}| \, | \cos(-2y) + i \sin(-2y) | < 1$

$\Rightarrow |e^{-2x}| \, | \cos^2(-2y) + \sin^2(-2y) | < 1$

$\Rightarrow |e^{-2x}| < 1$

$\Rightarrow e^{-2x} < 1 \quad \because e^{-2x} > 0 \; \forall x \in \mathbb{R}$

$\Rightarrow \frac{1}{e^{2x}} < 1 \Rightarrow 1 < e^{2x} \Rightarrow e^{2x} > 1$

$\Rightarrow e^{2x} > e^{0}$

$e^{2x}$ is $\uparrow$ function

$\Rightarrow 2x > 0 \Rightarrow x > 0 \Rightarrow \boxed{Re z > 0}$

Now Conversely we have given 'q' & we prove 'p'

$q \Rightarrow Re z > 0 \Rightarrow x > 0$

$\therefore |e^{-2z}| = |e^{-2(x+iy)}|$

$= |e^{-2x-2iy}|$

$= |e^{-2x} \cdot e^{-2iy}|$

$|e^{-2z}| = e^{-2x} < 1 \quad , \quad x > 0$

$Re z > 0$

**Question :-**

Examine the behavior of

$e^{x+iy}$ as $x \to -\infty$

$e^{x+iy}$ as $y \to +\infty$

Case :- $e^{x+iy}$ $x \to -\infty$

$\lim_{x \to -\infty} e^{x} \cdot e^{iy}$

$= (\lim_{x \to +\infty} e^{x}) e^{iy}$

$= (0) e^{iy} = 0$

$e^{z} \to 0$, whenever $x \to -\infty$

Case :- $\lim_{y \to \infty} e^{z} = \lim_{y \to \infty} e^{x+iy}$

$= \lim_{y \to \infty} e^{x} \cdot e^{iy} \Rightarrow e^{x} (\lim_{y \to \infty} e^{iy})$

$$= e^{x} \lim_{y \to \infty} (\cos y + i \sin y)$$

Applying limit :-

$$= e^{x} [\underbrace{\cos \infty + i (\infty)}_{\text{Circular motion.}}]$$

$\cancel{= e^{x}}$

$e^{z}$ rotates over & over again about origion having radius. $e^{x}$